The Mathematics of Epidemic Modeling

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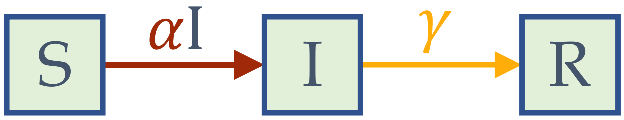
The **Variables**

All individuals in the population are classified as **S**usceptible, **I**nfectious, or **R**emoved

Susceptible individuals may become infectious when they interact with Infectious individuals



Infectious individuals are removed from the population over time

Removed individuals are no longer infectious, or no longer interacting with others, and cannot be reinfected

The **Parameters**

α The rate at which new infections occur. Can be written $\alpha = p \cdot n$, where n is the number of interactions a typical person has per day, and p is the probability that each interaction is sufficient to infect a susceptible individual.

γ The rate at which infectious individuals leave the population. The average length of time an individual is infectious is given by $1/\gamma$.

The **Differential Equations**

$$S'(t) = -\alpha \cdot I \cdot S$$

$$I'(t) = \alpha \cdot I \cdot S - \gamma \cdot I$$

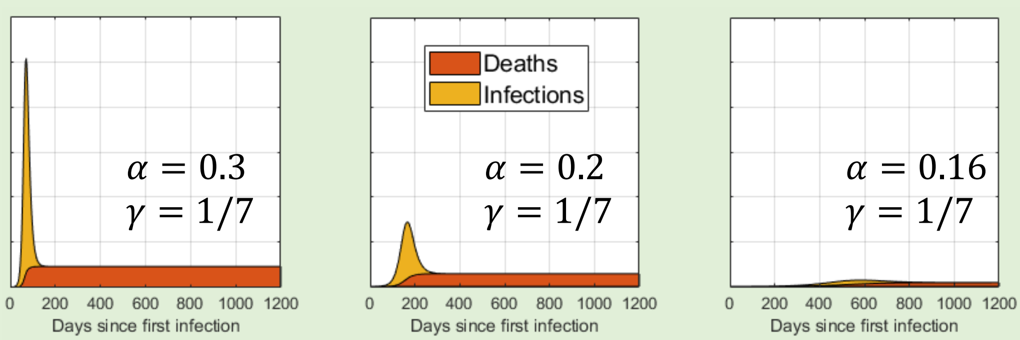
$$R'(t) = \gamma \cdot I$$

These differential equations describe how the number of people in each state **changes over time** (see reverse). The number of Susceptibles decreases, and the number of Infectious people increases, at a rate proportional to the fraction of the population that is infectious, **I**. Meanwhile, the infectious population also decreases as individuals recover, self-isolate, and are hospitalized.

The **Curve**

Decreasing α and increasing γ by *reducing interactions* and *lowering the chance of infection*

decreases the total number of cases, **slows the spread**, and **saves lives**



Why **Differential Equations Models?**

Although what we're interested in is often static: the peak number of infections, the final number of people who become ill, the length of the epidemic, and so on, the best way to predict those things is often by simulating the disease's spread over time using basic principles of epidemiology.

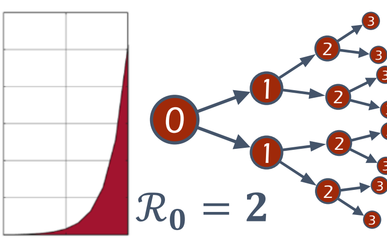
A differential equation is a description of how something changes over time. The symbols $\mathbf{S}'(t)$, $\mathbf{I}'(t)$, and $\mathbf{R}'(t)$ stand for the *rate of change of the variables* $\mathbf{S}$, $\mathbf{I}$, and $\mathbf{R}$. For instance, the $\mathbf{S}'(t)$ equation is: $\mathbf{S}'(t) = -\alpha \cdot \mathbf{S} \cdot \mathbf{I}$. This means: *The number of susceptible individuals decreases at rate* $\alpha \cdot \mathbf{S} \cdot \mathbf{I}$. Note that the rate of decrease depends on the sizes of *both* $\mathbf{S}$ and $\mathbf{I}$ – in a differential equations model each variable can affect the rate at which the other variables change, making such models a natural choice for thinking about all kinds of natural phenomena.

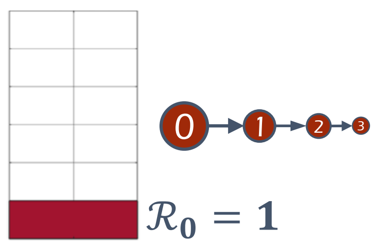
The **Basic Reproduction Number $\mathcal{R}_0$**

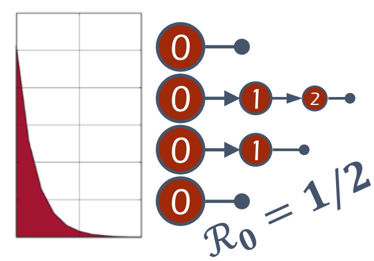
This number tells you how many **new** infections, on average, will be generated by a **single** infectious individual. It is equal to the Rate of new infections per day per individual (α) multiplied by the Length of time an individual is infectious for ($1/\gamma$)

$$\mathcal{R}_0 = \alpha/\gamma$$

If $\mathcal{R}_0$ is larger than one, the number of infections will **increase**. Therefore the main goal of public health policy is to **decrease $\mathcal{R}_0$ below 1**







How **Our Actions Matter**

Our goal as a community must be to help reduce $\mathcal{R}_0$ below 1, or equivalently, to make $\alpha < \gamma$.

Epidemic
$\alpha > \gamma$

Everyday actions like
- *social distancing*
- *frequently washing hands*
- *wearing cloth face masks when out*
- *disinfecting surfaces*

help **limit the number of interactions** an individual has **n** and **lowers the probability p that this person can infect others,** and thereby **decreasing α**

Decreasing α allows the healthcare system **time**, **energy** and **resources** to work at its best

Other ways to increase γ include:
- Quick and conscientious self-isolation
- Properly protecting health care professionals
- Widespread testing and contact tracing

Goal
$\alpha < \gamma$

Note: There is a time delay between the model's predictions and real-time information, due to the relatively long incubation and infectivity periods. In other words, patients who are being tested, isolated, and treated – today – are the result of transmissions that happened as much as two weeks ago. This makes it especially important to **remain vigilant** and **keep flattening the curve**.